Method for calculation of non-Gaussian generating functions in lattice models

1981 J. Phys. A: Math. Gen. 14 L239

(http://iopscience.iop.org/0305-4470/14/7/003)

J. Phys. A: Math. Gen. 14 (1981) L239–L241. Printed in Great Britain

LETTER TO THE EDITOR

Method for calculation of non-Gaussian generating functions in lattice models

W Jeżewski

Institute of Molecular Physics, Polish Academy of Sciences, 60–179 Poznań, Poland

Received 7 April 1981

Abstract. A method for the calculation of generating functions with non-Gaussian distributions is introduced. The method uses the Padé approximant technique for series of Stieltjes.

The application of the perturbation expansions in field theory and statistical mechanics is usually based on the use of a Gaussian model as the zeroth-order approximation. There has been a great deal of interest in studying alternative expansions which employ non-Gaussian models as the zeroth-order approximation (Wortis 1974, Wilson 1974, Kogut and Susskind 1975, Tamvakis and Guralnik 1978, Constantinescu 1980, Bender *et al* 1980). The reason for considering non-Gaussian expansions is that they appear to provide a new insight into the structure of theory and to be useful for calculational purposes.

Were we introduce an approximation method which applies to the lattice theory with non-Gaussian distributions. At first consider the integral

$$Z_1(g)=\int_0^{+\infty} \mathrm{d}x\, \exp(-gx-x^2) \tag{1}$$

with g being an arbitrary complex parameter such that $|g|<\infty$ and Re $g>0$. We wish to express $Z_1(g)$ by series of Stieltjes and subsequently to apply the Padé approximants to this series. To do this we express the function $Z_1(g)$ as a power series in g:

$$Z_1(g)=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\frac{1}{2}+\frac{1}{2}k)}{k!}g^k. \tag{2}$$

Let us represent the above relation in the following form

$$Z_1(g)=(2\pi\mathrm{i})^{-1}\int_{\mathrm{C}} \mathrm{d}x s(x)\, \mathrm{e}^{-gx} \tag{3}$$

where C is a contour encircling the positive real axis. Using the relation (Abramowitz and Stegun 1965)

$$\frac{1}{\Gamma(z)}=-(2\pi\mathrm{i})^{-1}\int_{\mathrm{C}} \mathrm{d}t(-t)^z\, \mathrm{e}^{-t} \qquad |z|<\infty \tag{4}$$

0305-4470/81/070239+03\$01.50

one can easily check that the quantity $s(x)$ is given by

$$s(x)=\frac{1}{2}\sum_{k=0}^{\infty} c_k x^{-(k+1)} \tag{5}$$

with

$$c_k=\Gamma(\tfrac{1}{2}+\tfrac{1}{2}k)=\int_0^{+\infty} u^{k/2}\,\mathrm{d}\phi(u) \tag{6}$$

$$\mathrm{d}\phi(u)=u^{-1/2}\mathrm{e}^{-u}\,\mathrm{d}u. \tag{7}$$

According to equation (7) the function $\phi(u)$ can be assumed to be

$$\phi(u)=\int_0^u \mathrm{d}y\, y^{-1/2}\,\mathrm{e}^{-y}. \tag{8}$$

This function is bounded and non-decreasing in the interval $0\leqslant u<\infty$, and takes on infinitely many values in this interval. Thus the series $s(x)=\frac{1}{2}\sum_{k=0}^{\infty} c_k x^{-k}$ is a series of Stieltjes (Baker 1970). It follows from the convergence theorems (Baker 1970) for series of Stieltjes that any sequence of $[N, N+j](-x^{-1})$, $j\geqslant -1$, Padé approximants for the series $s(x)$ converges to an analytic function for $\mathrm{Re}\, x<|x|$. The convergence is uniform with respect to x in any compact region in the complex x plane cut along the positive x axis. We may write the $[N, N+j](-x^{-1})$ Padé approximants to the series $s(x)$ in the form

$$[N, N+j](-x^{-1})=\sum_{k=1}^{N}\frac{\alpha_{k,N}}{1-\beta_{k,N}x^{-1}}+\sum_{l=0}^{j}\gamma_{l,N}(-x^{-1})^l, \qquad j\geqslant 1. \tag{9}$$

It is well known that for series of Stieltjes (Baker 1970)

$$c_k,\ \alpha_{k,N},\ \beta_{k,N}>0.$$

We now perform the integral of equation (3) with $s(x)$ expressed by Padé approximants. Inserting equation (5) into equation (3), using equation (9) and applying the Cauchy integral theorem one finds

$$Z_1^{(N,j)}(g)=\frac{1}{2}\left(\sum_{k=1}^{N}\alpha_{k,N}\exp(-g\beta_{k,N})+\sum_{l=0}^{j}\gamma_{l,N}\frac{g^l}{l!}\right), \qquad j<\infty. \tag{10}$$

Let us discuss the problem of the convergence of the sequence $Z_1^{(N,j)}(g)$ for N tending to infinity. We note that the function $f(g, x)=\exp(-gx)$ is regular in a uniform neighbourhood of the positive, real x axis and $(\ln x)^{1+\eta}f(g, x)\to 0$ as $x\to\infty$ for some $\eta>0$. Consequently, the $Z_1^{(N,j)}(g)$ converge in the limit as N tends to infinity (Baker 1970).

Using the result of equation (10) we can calculate the following integral

$$Z(a)=\int_{-\infty}^{+\infty}\mathrm{d}x\,\exp(ax-x^2) \tag{11}$$

with a being an arbitrary complex parameter such that $|a|<\infty$. Then we have

$$\begin{aligned} Z^{(N,j)}(a)&=[1+\exp(-2a\,\mathrm{d}/\mathrm{d}g)]Z_1^{(N,j)}(g)|_{g=a}\\ &=\sum_{k=1}^{N}\alpha_{k,N}\cosh(\alpha\beta_{k,N})+\sum_{i=0}^{j}\gamma_{l,N}\frac{a^{2l}}{(2l)!} \qquad j<\infty. \end{aligned} \tag{12}$$

Consider now the generating function

$$W(a) = \ln Z(a; a_3, a_5, \ldots, a_m; b_2, b_4, \ldots, b_n) \tag{13}$$

with

$$Z(a; a_3, a_5, \ldots, a_m; b_2, b_4, \ldots, b_n) \equiv Z$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(ax + \sum_{i=1}^{m} a_{2i+1}x^{2i+1} + \sum_{i=1}^{n} b_{2i}x^{2i}\right) \tag{14}$$

where a, a_i, b_i are arbitrary, in general, complex parameters such that $|a|$, $|a_i|$, $|b_i| < \infty$, and $n > m$, Re $b_n < 0$, so that the integral of equation (14) is convergent. This integral can be written in the form

$$Z = \exp\left(\sum_{i=1}^{m} a_{2i+1} \frac{\mathrm{d}^{2i+1}}{\mathrm{d}a^{2i+1}} + (b_2+1)\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \sum_{i=1}^{n} b_{2i+2} \frac{\mathrm{d}^{2i+2}}{\mathrm{d}a^{2i+2}}\right) Z(a). \tag{15}$$

In order to calculate Z, we use the approximation of equation (12). Substituting this equation into equation (15) and restricting ourselves to the case of the $[N, N-1]$ and $[N, N]$ Padé approximants we find, after some algebra, that the generating function $W(a)$ is given by

$$\exp[W(a)] = \gamma_{0,N} + \sum_{k=1}^{N} \alpha_{k,N} \cosh\left(\alpha\beta_{k,N} + \sum_{i=1}^{m} \alpha_{2i+1}\beta_{k,N}^{2i+1}\right)$$
$$\times \exp\left((b_2+1)\beta_{k,N}^2 + \sum_{i=1}^{n} b_{2i+2}\,\beta_{k,N}^{2i+2}\right) \tag{16}$$

where the first component is absent in the case of the $[N, N-1]$ Padé approximants. It should be pointed out that the sum in equation (16) is a well defined finite quantity for all N (including $N = \infty$) whenever $n > m$ and Re $b_n < 0$.

In conclusion, we have introduced a simple method for calculating the generating functions with non-Gaussian distributions. The generating function calculated here may be of use in a variety of problems concerning the non-Gaussian lattice models.

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)

Baker G A 1970 *The Padé Approximants in Theoretical Physics* ed G A Baker and J L Gammel (New York: Academic)

Bender C M, Cooper F, Guralnik G S, Moreno H, Roskies R and Sharp D H 1980 *Phys. Rev. Lett.* **45** 501

Constantinescu F 1980 *J. Math. Phys.* **21** 2278

Kogut J and Susskind L 1975 *Phys. Rev.* D **11** 395

Tamvakis K and Guralnik G S 1978 *Nucl. Phys.* B **146** 224

Wilson K G 1974 *Phys. Rev.* D **10** 2445

Wortis M 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (London: Academic)